Probability 2nd Edition: Now with Extra Beans!

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Fundamental Counting Principle

FUNDAMENTAL COUNTING PRINCIPLE

Suppose that two events occur in order. If the first can occur in m ways and the second in n ways (after the first has occurred), then the two events can occur in order in $m \times n$ ways.

Factorials!!!!!!!

n! = n * (n-1) * (n-2) * (n-3) * ... *1

6! = 6*5*4*3*2*1 = 720

EXAMPLE 3 ■ Using Factorial Notation

In how many different ways can a race with six runners be completed? Assume there is no tie.

SOLUTION

There are six possible choices for first place, five choices for second place (since only five runners are left after first place has been decided), four choices for third place, and so on. So, by the Fundamental Counting Principle, the number of different ways this race can be completed is

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720 \qquad \blacksquare$$

Permutations and Combinations

PERMUTATIONS OF n OBJECTS TAKEN r AT A TIME

The number of permutations of n objects taken r at a time is

$$P(n, r) = \frac{n!}{(n-r)!}$$

COMBINATIONS OF n OBJECTS TAKEN r AT A TIME

The number of combinations of n objects taken r at a time is

$$C(n, r) = \frac{n!}{r!\,(n-r)!}$$

Distinguishable Permutations

DISTINGUISHABLE PERMUTATIONS

If a set of n objects consists of k different kinds of objects with n_1 objects of the first kind, n_2 objects of the second kind, n_3 objects of the third kind, and so on, where $n_1 + n_2 + \cdots + n_k = n$, then the number of distinguishable permutations of these objects is

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_k!}$$

Probability of an Event

DEFINITION OF AN EVENT

If S is the sample space of an experiment, then an **event** is any subset of the sample space.

DEFINITION OF PROBABILITY

Let S be the sample space of an experiment in which all outcomes are equally likely, and let E be an event. The probability of E, written $P(E)$, is

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{number of elements in } E}{\text{number of elements in } S}$$

Complement of an Event

The **complement** of an event E is the set of outcomes in the sample space that is not in E. We denote the complement of an event E by E'. We can calculate the probability of E' using the definition and the fact that $n(E') = n(S) - n(E)$:

$$P(E') = \frac{n(E')}{n(S)} = \frac{n(S) - n(E)}{n(S)} = \frac{n(S)}{n(S)} - \frac{n(E)}{n(S)} = 1 - P(E)$$

PROBABILITY OF THE COMPLEMENT OF AN EVENT

Let S be the sample space of an experiment and E an event. Then

$$P(E') = 1 - P(E)$$

This is an extremely useful result, since it is often difficult to calculate the probability of an event E but easy to find the probability of E', from which $P(E)$ can be calculated immediately using this formula.

(Dependent) Mutually Exclusive Events

Mutually exclusive events are events that have no common possible outcomes between them.

PROBABILITY OF THE UNION OF MUTUALLY EXCLUSIVE EVENTS

If E and F are mutually exclusive events in a sample space S, then the probability of E *or* F is

$$P(E \cup F) = P(E) + P(F)$$

(Dependent) Non-Mutually Exclusive Events

PROBABILITY OF THE UNION OF TWO EVENTS

If E and F are events in a sample space S, then the probability of E *or* F is

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Independent Events

PROBABILITY OF THE INTERSECTION OF INDEPENDENT EVENTS

If E and F are independent events in a sample space S, then the probability of *E and F* is

$$P(E \cap F) = P(E)P(F)$$

Expected Value

DEFINITION OF EXPECTED VALUE

A game gives payoffs $a_1, a_2, \ldots, a_n$ with probabilities $p_1, p_2, \ldots, p_n$. The **expected value** (or **expectation**) E of this game is

$$E = a_1p_1 + a_2p_2 + \cdots + a_np_n$$

Problems! (The Fun Part)

A Trip to the Bean Store

Today, Arnex is taking a break from his job at the Angle Sorting Factory to go visit the new store that just opened, Bernard's Bean Bonanza!! He is hoping to buy some Lima Beans, but the store only sells them on Thursdays when it's raining; however Arnex has forgotten his combo pocket calendar and universal tuning fork, so he doesn't know what weekday it is, and it is equally likely to be any of the five. He does, however, know from the weather forecast he heard this morning in his head that there is a 40% chance of rain. How likely is it that Arnex will be able to buy the Lima Beans today?

A Trip to the Bean Store (Solution)

It is a weekday, so there is a ⅕ chance that it is a Thursday. There is a 40%, or ⅖ chance of rain.

So ⅕ * ⅖ = 2/25.

There is a 2/25 chance that Arnex can buy the Lima Beans today (assuming he remembered to bring money). Better luck next time, Arnex!

A Delicious Three-Bean Salad [Part 1]

When Arnex arrives inside of Bernard's Bean Bonanza!! he sees the calendar and realizes it is Wednesday, so he can't get his Lima Beans; however, he notices that today's special is Bernard's famous three-bean salad. The salad is served with a choice of one of three lettuces, one of two varieties of onions, one of six dressings, and three of six varieties of beans. Bernard advertises that they serve over 700 unique salads, but Bernard has been known to inflate his numbers in order to sell more product, as well as to increase his chances to win a Beanie Award, the awards given to high quality bean vendors. Is Bernard's advertisement of 700 salads accurate, or is he just trying to get that coveted Beanie again?

A Delicious Three-Bean Salad [Part 1] (Solution)

This is a fundamental counting principle problem.

3 lettuces * 2 onions * 6 dressings * 3 of 6 beans =

$3 * 2 * 6 * C(6, 3) = 36 * 6!/(3!3!) = 720$

Bernard sells 720 different salads. It turns out he's telling the truth for once!

A Delicious Three-Bean Salad [Part 2]

Enticed by the sheer variety given by the salads, Arnex decides he would like to order one. To his dismay, he discovers that while you get to choose what type of lettuce, onion, and dressing you receive, you cannot decide which beans are in your salad. Instead, the beans are chosen randomly by a computer. Additionally, the pricing of the salad is based upon which beans are in it. There are six available beans: Garbanzo ($3), Kidney ($2), Pinto ($2.50), Coffee ($3.50), Black ($1.50), and Jelly Beans ($8). Each bean is equally likely to be chosen by the computer. Because of this element of randomness, Arnex would like to determine how much the salad is most likely to cost him, to be sure he can afford it. What is the average expected value of the salad?

A Delicious Three-Bean Salad [Part 2] (Solution)

Because 3 of the 6 beans must be picked, each bean has a ½ chance of being in the salad. Viewing this as an expected value problem, each price is multiplied by the likelihood of it happening, so each price is multiplied by ½. The prices are then all added together to find the expected value.

½*3 + ½*2 + ½*2.5 + ½*3.5 + ½*1.5 + ½*8 =

1.5 + 1 + 1.25 + 1.75 + .75 + 4 = 10.25

The salad is expected to be $10.25, a bargain for such a high quality meal.

A New Face in the Store

Meanwhile, as Arnex is pondering the price of his prospective three-bean salad, local town troublemaker Rex Tangle enters the store. He has heard that the store was hosting a state-sponsored bean lottery game, and Rex hopes to win big. The game consists of a bowl of 12 beans: 3 Garbanzo, 3 Coffee, 3 Pinto, and 3 Mortgage Lifter Beans. The game costs $2 to play, and the player grabs 3 beans randomly out of the bowl without looking. If they happen to grab 3 beans of the same variety, they win $100. Rex plans to spend $200 playing this game, and to hopefully make even more money as a result. What is the expected value of this game, and does Rex's plan work?

A New Face in the Store (Solution)

First we must calculate the chances of a win. There are only 4 different draws that win (one for each draw of 3 matching beans), and there are C(12, 3) possible draws total.

Therefore, the likelihood of a win is 4/C(12, 3) = 4/220 = 1/55. This means the likelihood of a loss is 54/55.

The monetary result of a win is $100 prize money - $2 playing charge = $98. The monetary result of a loss is -$2.

So the expression of expected value is 1/55 * 98 - 54/55 * 2 = -$0.18

Rex will gradually lose money as he plays the lottery game.

A Brief Intermission (Guest Problems)

A True Artist

Ezra is a sculptural artist of much acclaim. When he has a vision, it always leads to an inspired installation. His latest piece, *The Beanch*, is a park bench made entirely of dehydrated refried beans. The sculpture was quickly purchased by The New York Metropolitan Tri-State Area Museum of High Quality Art and Art-Like Objects. Ezra has just realized that he forgot to tell the museum that *The Beanch* will fall apart when placed in the rain and he is worried that they will place the stunning installation outdoors. The Museum has 24 rooms: 8 large and 16 small. Of these rooms, 2 large and 6 small are outdoors. Assuming the museum will put the piece in a random room with the larger rooms having twice the weight of the smaller rooms, what is the likelihood that Ezra's sculpture will remain intact so it can receive the glory it deserves in its new home.

A True Artist (Solution)

To find the probability, let's alter the numbers to reflect the weights of the room sizes. Let's replace the 8 large rooms with 16 small rooms so everything has equal weight. Because of this increase in rooms, the denominator will be 32. Now for the numerator. 2 of the large rooms were outside, so we'll bump that up to 4. The 6 other outdoor rooms are fine as is. This leaves us with a total number of outdoor rooms of 10. Therefor, there are 22 indoor rooms.

$$\frac{22}{32}$$

There is a 68.75% chance that *The Beanch* will be safely housed indoors where it can be worshiped as the totem of human achievement that it is.

Baked Beans

Larry is baking a birthday cake for his now 4-year old son, Harry. Making the batter, he reaches step 7 on the recipe: add 5 beans. Larry had not realised the cake called for beans, but luckily he had a pouch of 10 beans in the pantry for just such an occasion. He gingerly opens the small bag and begins counting out beans. 1, 2, — Oh no! After putting only 2 beans into the batter, he drops the pouch, beans spilling out everywhere. Assuming each bean that spills has a 50% chance of falling into the batter, what are the odds that the batter receives the 5 beans it requires and nothing more? Larry would be devastated if Harry's cake turned out wrong.

Baked Beans (Solution)

There are 10 beans in the pouch. The recipe calls for 5, and 2 are already in. That means there are 8 falling from the pouch, and 3 more are needed in the cake. There are 2^8 ways in which the beans can fall. There are C(8, 3) ways that exactly 3 can be chosen to fall in. So P(E) =

$$\frac{{}_{8}C_{3}}{2^{8}}$$

= 56/256 = 7/32

The probability is 7/32.

Larry can rest easy knowing that Harry's birthday is not ruined.

Now, Back to Our Story

We've bean in suspense for too long.

Lodging a Complaint

After Rex has used all his money on the lottery game and run out, he realizes that the system has cheated him out of his money. Enraged, he goes to the counter at the front desk to lodge a complaint. Unfortunately, he failed to realize that in order to lodge a complaint, patrons must first win a game of musical bean bag chairs. Also unfortunately, one of the other people playing is his old nemesis, Johann Perkins, the pharmacist. Johann is a ruthless player, who will edge anyone out of a win in musical bean bag chairs. To increase his chances of winning, Rex decides he must be sure not to sit next to Johann. If there are 7 bean bag chairs arranged in a line, with 7 players including Rex and Johann, how many ways can the players sit to start the game, if Rex will not sit with Johann?

Lodging a Complaint (Solution)

In problems like this, we find the number of arrangements of the people that don’t have special conditions there can be, and multiply that by the number of ways we can insert the special people into that group while still following the rules.

Removing Johann and Rex from the group we are just arranging 5 people, so they can be arranged 5! Ways. Then, once they are arranged, there are six spots that either Johann or Rex can be put in, and we choose two of them, so there are $P(6, 2)$ ways to insert them into the group. $5! * P(6, 2) = 3600$.

There are 3600 arrangements in which Rex can avoid Johann.

The Sting of Rejection

After 3 rounds of musical bean bag chairs, Rex finally wins a game and proceeds to the complaints counter. He takes one of the pieces of paper made from bean skin that they provide and writes a complaint on it about how the game took all his money and cheated him. After reading the complaint the counter clerk remains silent, places the slip of bean skin paper in a blender containing many other slips, and makes them into a delicious bean skin smoothie. Infuriated about how flagrantly his complaint has been ignored, Rex begins to rampage around the store, destroying everything in his path. This activates the store's Ballistic Energy Analysis to Neatly Present Options for Liable Employees system (or BEANPOLE system for short). The system finds that there is a 57% chance that the store's bean powered reactor will be hit in Rex's rampage, and that it has a 39% chance of an explosive catastrophic failure if hit. How likely is an explosion from the bean reactor?

The Sting of Rejection (Solution)

The reactor has a 57% chance of being hit and a 39% chance of catastrophic failure.

57% * 39% = 22.23%

There is a 22.23% chance that Rex's rampage around the store will cause a catastrophic failure that destroys the entire store.

A Hero Rises

Luckily for the patrons of the store, Arnex happens to be just finishing up his three-bean salad order as Rex begins destroying the store. Right next to the deli counter where he's standing is the weapons counter, which stores all of the store's weapons of self defense. As soon as the rampage begins, he dives behind the counter and grabs onto the first weapon he can: a medium caliber bean gun. The bean gun has a cartridge that stores 8 Mexican Jumping Beans, which is admittedly below the standard capacity of a bean gun of its power, but he'll just have to make do. Unfortunately, due to design flaws in the latest model of the BeanBlaster 2000, the gun has a terrible accuracy rating; however, due to Arnex's extensive expertise in angle sorting, he is able to quickly pull an angle out of his pocket and affix it to the barrel of the gun, significantly increasing its accuracy. A bean fired from the gun will now have a 50% hit chance on any target. Arnex also remembers from his Angle Security and Safety Training that he received at the start of his job at the Angle Sorting Factory that it takes about 6 hits from a Mexican Jumping Bean rifle to take down an adversary. Knowing this, what are the chances that Arnex will be able to take down Rex with his magazine of 8 beans?

A Hero Rises (Solution)

This is an at least problem. We must find the number of combinations of shots there are in which 6 or more hit, and divide them the total number of possible hit/miss records. Because the chance for any one shot is 50/50, each possible combination of hits and misses is equally likely.

There are C(8, 6) ways in which 6 can hit, C(8, 7) ways in which 7 can hit, and 1 way in which all can hit. Then there are 2^8 ways for any number to hit because of the fundamental counting principle.

(C(8, 6) + C(8, 7) + 1)/2^8 = 37/256 = 14.5%

Arnex has a 14.5% chance of taking down Rex with the bean gun. It's a long shot, but he has to take it, for the good of everyone.

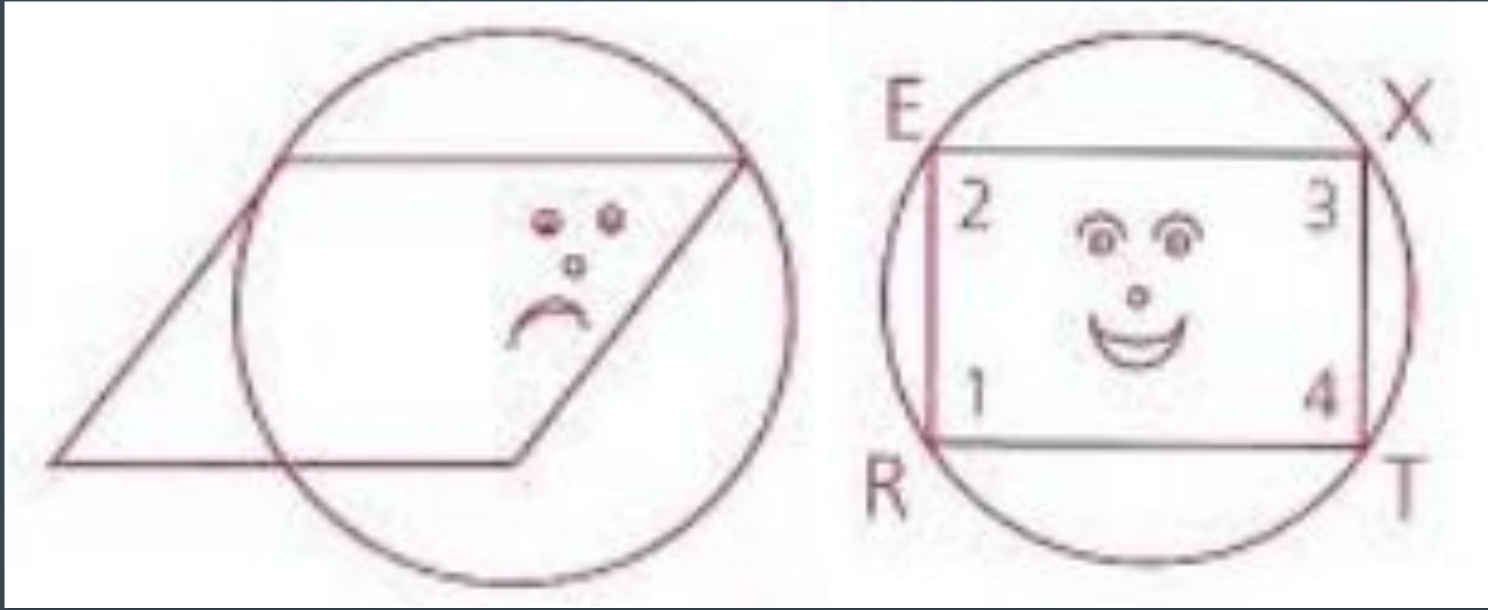
A Sudden Intervention

Arnex begins to fire the bean gun at Rex. He makes the first two shots, but misses the third. He makes the fourth, fifth, and sixth shots, and misses the seventh. With only one shot left, he realizes it has come down to this. He aims and steadies his hand. But suddenly, a flying can of Billy Roberts' Famous Quintuple Refried Bean Composite hits him in the hand, and he drops the gun. It seems as though there is no hope, when suddenly a circular piece of the ceiling falls and hits Rex on the head, knocking him out cold. As sunlight streams in, on the roof Arnex can see a black government helicopter, its blades still whirring down. It must have come for an unscheduled covert retail security assessment, and in their entrance the agents from an unknown government agency ended up dropping the ceiling piece on Rex. Whether by luck, or by semi-divine government intervention, the store has been saved. As the government agents drag Rex away, the BEANPOLE System finds that Rex has a 27% chance of escaping from the handcuffs that the government agents have hastily assembled out of loose driftwood and untreated bean paste. The agents also have a 84% success record when catching escaped prisoners. What are the chances that Rex will escape from the custody of the unidentified government agency?

A Sudden Intervention (Solution)

There is a 27% chance that Rex escapes the handcuffs. There is an 84% chance the agents would catch him if he did, which means there is a 100%-84% = 16% chance they won't catch him. 27% * 16% = 4.32%. There is a 4.32% chance that Rex will escape the authorities. Luckily though, he does not escape, and the agents successfully apprehend him and put him in prison indefinitely. Now endowed with a sense of a job well done (though wrongly so; he did nothing), Arnex went back to the deli counter to relax and eat his three bean salad with Coffee, Garbanzo, and Jelly Beans.

Appendix A: Rex’s Mugshot



Appendix B: Unused Bean Ideas

- Build-a-Bean Workshop
- A child named Bean

Appendix C: Appendicitis

Appendix D: Classic Problems

Permutation

In the 2017 lunar fall olympics, there are 8 contestants in the gourd carving n' tossing event. How many ways can a gold silver and bronze award be given out among them?

Solution

Since there are three different awards this is a permutation because order matters

The formulae for permutations: $\frac{n!}{(n-k)!}$

Since there are eight contestants it would be: $\frac{8!}{(8-3)!}$

This equals 336.

Com - (recycling) bin - ations

Because of a lack of funding for the 2017 lunar fall olympics, instead of gold, silver and bronze awards, three recycled tin cans will be given out. Now how many *combinations* of the 8 contestants will be utterly disappointed by their recycled tin can?

Solution Time!

Since all three of the awards are the same, order does not matter in the outcome of who got which award, so this is a combination.

The formula for combinations is: $C(n,k) = \frac{n!}{(n-k)!k!}$

Given the 8 people and the 3 awards: 8! / (8-3)!3! = 56

Grand Master Wheel of Super Fortune

This is it: the big day. You've saved your money. You've made the trip. You're ready. Today is the day you play the Grand Master Wheel of Super Fortune at the Lucky Desert Rabbit Bouncy Castle and Casino Supercenter, in Las Vegas, Maine. The famed game features a spinning wheel with 31 slots. One of the slots is the Super mystery slot, which, if landed on, rewards you with an incredible fortune of 12 dollars. The game costs two dollars to play, and you've been saving your McDonald's wages for twelve years in preparation, so you've brought 1000 dollars with you. Are you going to win big, or have you made some kind of terrible mistake?

Solution

You gain \$9 with probability 1/12. You lose \$1 with probability 11/12. Therefore:

$E = (10)(1/31) + (-2)(30/31) = -1.61$

You manage to spend all your money and drive yourself into poverty.

Boring ~~Normal~~ Problems

Unfortunately, writing $^{\text{FUN}}$FUN$^{\text{FUN}}$ word problems takes some time. And while we certainly have more time than sense, we don't have quite THAT much more. But we do have a 20 problem requirement. Therefore, the rest of this presentation will be boring problems, devoid of life, excitement, or any sense of meaning. Thanks.

Boring ~~Normal~~ Problem #1

The Boring Club has 7 members. In how many ways can it choose a President, Vice President, and Secretary?

Boring ~~Normal~~ Problem #1 (Solution)

This is just a boring fundamental counting principle problem.

7*6*5 = 210

210 ways. woooooo

Boring ~~Normal~~ Problem #2

A boring true-false test contains 8 really boring questions. How many ways can the test be completed?

Boring ~~Normal~~ Problem #2 (Solution)

This is just a boring pizza-topping style problem.

2^8 = 256

256 ways. i’m on the edge of my seat here

Boring ~~Normal~~ Problem #3

How many four letter words can be made from the letters in the word “BORING”?

Boring ~~Normal~~ Problem #3 (Solution)

Nothing to see here but a basic, uninteresting permutation problem.

There are six letters in BORING. We choose four.

P(6, 4) = 6!/2! = 6*5*4*3 = 360

360 words. isn’t this exciting

Boring ~~Normal~~ Problem #4

How many ways can 4 boring people be chosen from a unimaginative group of 10 extremely dull people?

Boring ~~Normal~~ Problem #4 (Solution)

A really rather mundane combination problem.

C(10, 4) = 10!/(4!6!) = (10*9*8*7)/(4*3*2*1) = 10*3*7 = 210

210 ways. what a ruthless plot twist

Boring ~~Normal~~ Problem #5

An uninteresting letter is randomly chosen from the phrase “ABSOLUTE LIFELESSNESS”. What is the probability that the letter chosen is not an “S”?

Boring ~~Normal~~ Problem #5 (Solution)

This is a drab unoriginal probability problem. There are 20 letters. 15 of them are not "S" or whatever.

$P(E) = 15/20 = 3/4$

¾ probability. never woulda seen that coming

Boring ~~Normal~~ Problem #6

A really platitudinous roulette table or something has 38 slots. One is 0, one is 00, and the rest are marked 1-36. What is the probability of landing on a frighteningly boring odd number?

Boring ~~Normal~~ Problem #6 (Solution)

An impossibly plebeian probability question. There are 38 slots. Uninterestingly, 18 of them have odd numbers.

P(E) = 18/38 = 9/19

9/19 probability. what a new and novel concept that was

Boring ~~Normal~~ Problem #7

An unexciting man pays $1 to roll a quite culturally irrelevant six sided die. If he rolls a six, he wins $5. Find the expected value, but really why even bother with this uncreative garbage?

Boring ~~Normal~~ Problem #7 (Solution)

An uneventful expected value problem. ⅙ chance to roll a six, ⅚ to not. +$4 if rolled a 6, -$1 if not.

$E = \frac{1}{6} * 4 + \frac{5}{6} * -1 = -\$0.17$

He loses $0.17 every play. what an unexpected surprise

Boring ~~Normal~~ Problem #8

A bag contains 6 frankly characterless marbles, 4 light grey and 2 dark grey. An unremarkable person of some kind chooses a marble, and if they choose a dark grey marble they win $3. What is the expectation of this game, assuming it even matters?

Boring ~~Normal~~ Problem #8 (Solution)

Just another extremely commonplace expected value problem. ⅓ chance to choose dark grey. +$3 if dark grey is chosen.

$E = \frac{1}{3} * 3 = \$1$

$1 expectation. Like anyone cares

Boring ~~Normal~~ Problem #9

An astonishingly boring class of 10 very uninteresting students are lining up to take an uneventful class photo. How many ways can the class be arranged?

Boring ~~Normal~~ Problem #9 (Solution)

This is just another completely pointless permutation problem.

P(10, 10) = 10! = 3,628,800

3,628,800 ways. incredible

THE BEAN DIP

Congrats! You made it past all the boring ~~normal~~ problems. Now you can go and live your interesting lives again. Have fun!